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Static Displacement Reanalysis of Modified Structures Using the Epsilon Algorithm

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I. Introduction

IN THE optimization of structural systems, it is very important to compute the static displacements of the structures when the parameters of the structures are changed. One of the main obstacles is the high computational cost involved in the solution of large-scale problems. Therefore, the reanalysis problems excite the interest of many researchers and both approximate and exact reanalysis methods have been reported and reviewed [1]. In general, the following factors are considered in choosing an approximate behavior model for a specific optimal design problem [2]. 1) the accuracy of the calculations or the quality of the approximations; 2) the computational effort involved or the efficiency of the method, and 3) the ease of implementation.

At present, the various approximations have been developed. Barthelemy, Kirsch, and Haftka et al. [3–5] developed the global approximation (also called multipoint approximations), such as polynomial fitting or reduced basis methods. These approximations are obtained by analyzing the structure at a number of design points, and they are valid for the whole design space. Local approximation is called single-point approximations, such as the Taylor series expansion or the binomial series expansion about a given point in the design space. Local approximations are based on information calculated at a single point. These methods are effective only in cases of small changes in parameters of the structures. For large changes in the design, the accuracy of the approximations often deteriorates, and they may become meaningless [6]. Kirsch [7–12] discussed the combined approximations, which attempt to give global qualities to local approximations.

Recently, the Padé approximation and Shanks transformation were used to improve the accuracy of the reanalysis [13,14]. It is shown that it can significantly improve the domain of convergence.

In this study, we use the epsilon algorithm [15–17] to deal with the static displacement reanalysis. The main objectives are to preserve the ease of implementation and to improve significantly the domain of convergence and the quality of the results, such that the method can be used in problems with very large changes in the structural parameters. A numerical example is demonstrated and the method is compared with the Kirsch combined approximation.

II. Epsilon Algorithm and Its Extension to the Vector Case

A. Epsilon Algorithm

We consider an infinite sequence $\{a_0, a_1, a_2, \ldots\}$, and let the s_n be the partial sum of the sequence; then we have a new sequence $\{s_0, s_1, s_2, \ldots\}$

$$s_n = \sum_{i=0}^{n} a_i \qquad n = 0, 1, 2, \ldots \tag{1}$$

For the sequence $\{s_0, s_1, s_2, \ldots\}$, we construct an iterative form

$$\varepsilon_{-1}^{(j)} = 0 \tag{2}$$

$$\varepsilon_{0}^{(j)} = s_j \tag{3}$$

$$\varepsilon_{k+1}^{(j)} = \varepsilon_{k-1}^{(j+1)} + [\varepsilon_{k}^{(j+1)} - \varepsilon_{k}^{(j)}]^{-1} \qquad j, k = 0, 1, 2, \ldots \tag{4}$$

To illustrate the computation of the iteration form, the epsilon algorithm is given in Table 1, where $n = 4$.

The epsilon algorithm was developed by Wynn for the acceleration of convergence of a sequence and it extends the convergence domain [15–17].

Example 1: Considering the function $f(x) = \ell n(1 + x)$, expanding it to series,

$$\ell n(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \tag{5}$$

We know the convergence domain of the right-hand side of Eq. (5) is $[-1, 1]$. The sequence of the partial sum is

$$s_n(x) = \sum_{k=1}^{n+1} (-1)^{k-1}\frac{x^k}{k} \qquad n = 0, 1, 2, \ldots \tag{6}$$

When $x = 1$, we get the sequence of the partial sum s_0, s_1, s_2, s_3, and s_4, and applying the epsilon algorithm as in Table 1, we get the epsilon iterative as in Table 2, where $\varepsilon_4^{(0)} = 0.693333$ is a good approximation to $\ell n(2) \approx 0.67314718$. If we use Eq. (6) to evaluate $\ell n(2)$, the computed effort must be to $n = 5368$, that is, the $s_{5368}(1)$ is near the $\varepsilon_4^{(0)} = 0.693333$.

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Table 1 The epsilon iterative table

0		0		0		0		0		0	$\cdots$
	s_0		s_1		s_2		s_3		s_4	$\cdots$	
		$\varepsilon_1^{(0)}$		$\varepsilon_1^{(1)}$		$\varepsilon_1^{(2)}$		$\varepsilon_1^{(3)}$	$\cdots$		
			$\varepsilon_2^{(0)}$		$\varepsilon_2^{(1)}$		$\varepsilon_2^{(2)}$	$\cdots$			
				$\varepsilon_3^{(0)}$		$\varepsilon_3^{(1)}$	$\cdots$				
					$\varepsilon_4^{(0)}$	$\cdots$					

When $x = 2$, we know the series in Eq. (5) is divergent. Using the epsilon algorithm in Eq. (6) as a formal series, we have Table 3. $\varepsilon_4^{(0)} = 1.1014$ is very close to the real value of $\ln(3) \approx 1.0986123$. From this example we can see that although the series is divergent, the epsilon algorithm can yield a sufficient result after fourth order iteration using the first five terms of the series.

From the example we can see that the even row of the epsilon algorithm has the property of accelerating the convergence of a sequence. It has been shown that the even row of the epsilon algorithm is the Shanks transformation [15]:

$$\varepsilon_{2k}^{(n)} = e_k(s_n), \qquad n, k = 0, 1, 2, \ldots \tag{7}$$

where $e_k(s_n)$ is called Shanks transformation that was defined in the form of a quotient of two Hankel determinants. In fact, the epsilon algorithm is a very practical way to construct Shanks transformation without the necessity of evaluating determinants [14–17]. Equation (7) is a fundamental result to show the connection between the epsilon algorithm and the Shanks transformation.

B. Extend the Epsilon Algorithm to the Vector Case

To extend the epsilon algorithm to the vector case is useful in the structure reanalysis [15]. Given a vector sequence $\{\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \ldots\}$, we construct the iteration form similar to Eqs. (2–4):

$$\boldsymbol{\varepsilon}_{-1}^{(j)} = \mathbf{0} \tag{8}$$

$$\boldsymbol{\varepsilon}_{0}^{(j)} = \mathbf{s}_j \tag{9}$$

$$\boldsymbol{\varepsilon}_{k+1}^{(j)} = \boldsymbol{\varepsilon}_{k-1}^{(j+1)} + [\boldsymbol{\varepsilon}_{k}^{(j+1)} - \boldsymbol{\varepsilon}_{k}^{(j)}]^{-1} \qquad j, k = 0, 1, 2, \ldots \tag{10}$$

The iteration formulas (8–10) are similar to Eqs. (2–4) for the scalar case except that they require the inverse of a vector. We defined

$$\mathbf{u}^{-1} = \frac{\mathbf{u}^*}{(\mathbf{u}^H\mathbf{u})} = \frac{\mathbf{u}^*}{\sum_{i=1}^{d} |u_i|^2} \tag{11}$$

where the asterisk denotes the complex conjugate and H is the Hermitian conjugate. The vector epsilon-algorithm table constructed by Eqs. (8–10), is similar to Table 1.

III. Matrix Perturbation and Combined Approximations for Static Displacement Analysis

The structure static displacement analysis equation is as follows:

$$\mathbf{K}_0\mathbf{u}_0 = \mathbf{f}_0 \tag{12}$$

where $\mathbf{K}_0$ and $\mathbf{f}_0$ are, respectively, the stiffness matrix and load vector of the finite element assemblage. This equation is known as the original problem in the following discussions. Assuming a change in design parameters, the problem becomes one of determining $\mathbf{u}$ when $\mathbf{K}_0$ and $\mathbf{f}_0$ are changed to $\mathbf{K}_0 + \boldsymbol{\Delta}\mathbf{K}$ and $\mathbf{f}_0 + \boldsymbol{\Delta}\mathbf{f}$, respectively. So, the static displacement analysis problem can be written as

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{13}$$

where

$$\mathbf{K} = \mathbf{K}_0 + \boldsymbol{\Delta}\mathbf{K} \tag{14}$$

$$\mathbf{f} = \mathbf{f}_0 + \boldsymbol{\Delta}\mathbf{f} \tag{15}$$

To solve Eq. (13), many methods are developed. Now, we introduce the matrix perturbation method. Assume

$$\mathbf{K} = \mathbf{K}_0 + \varepsilon\boldsymbol{\Delta}\mathbf{K} \tag{16}$$

$$\mathbf{f} = \mathbf{f}_0 + \varepsilon\boldsymbol{\Delta}\mathbf{f} \tag{17}$$

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \varepsilon^3\mathbf{u}_3 + \cdots \tag{18}$$

substituting Eqs. (16–18) into Eq. (13), compare the orders of the parameter ε, getting

$$\mathbf{u}_0 = \mathbf{K}_0^{-1}\mathbf{f}_0 \tag{19}$$

$$\mathbf{u}_1 = \mathbf{K}_0^{-1}(\boldsymbol{\Delta}\mathbf{f} - \boldsymbol{\Delta}\mathbf{K}\mathbf{u}_0) \tag{20}$$

$$\mathbf{u}_2 = -\mathbf{K}_0^{-1}\boldsymbol{\Delta}\mathbf{K}\mathbf{u}_1 \tag{21}$$

Table 2 The epsilon iterative table of Example 1 ($x = 1$)

	0		0		0		0	
$s_0 = 1$		$s_1 = 0.5$		$s_2 \approx 0.8333$		$s_3 \approx 0.5833$		$s_4 \approx 0.7833$
	−2		3		−4		5	$\cdots$
		0.7		0.6905		0.6944	$\cdots$	
			−102		248	$\cdots$		
				0.693333	$\cdots$			

Table 3 The epsilon iterative table of Example 1 ($x = 2$)

	0		0		0		0	
$s_0 = 2$		$s_1 = 0$		$s_2 \approx 2.6667$		$s_3 \approx -1.3333$		$s_4 \approx 5.0667$
	$-\frac{1}{2}$		$\frac{3}{8}$		$-\frac{1}{4}$		$\frac{5}{32}$	$\cdots$
		1.1429		1.0667		1.1282	$\cdots$	
			$-\frac{51}{4}$		16	$\cdots$		
				1.1014	$\cdots$			

$$\mathbf{u}_s = -\mathbf{K}_0^{-1}\Delta\mathbf{K}\mathbf{u}_{s-1} \qquad s = 2, 3, 4, \ldots \tag{22}$$

In fact, we can derivate Eqs. (19–22) in the following way:

$$(\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{u} = \mathbf{f}_0 + \Delta\mathbf{f}$$

$$\mathbf{u} = (\mathbf{I} + \mathbf{K}_0^{-1}\Delta\mathbf{K})^{-1}\mathbf{K}_0^{-1}(\mathbf{f}_0 + \Delta\mathbf{f})$$
$$= [\mathbf{I} - \mathbf{B} + \mathbf{B}^2 - \cdots + (-1)^k\mathbf{B}^k + \cdots]\mathbf{K}_0^{-1}(\mathbf{f}_0 + \Delta\mathbf{f}) \tag{23}$$

where $\mathbf{B} = \mathbf{K}_0^{-1}\Delta\mathbf{K}$. From Eq. (23), we can get the same solution as Eqs. (19–22). We call $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \cdots + \mathbf{u}_s$ the sth order perturbation solution.

In the Kirsch combined approximation, $\mathbf{u}$ can be approximated by the linear combination

$$\mathbf{u} = y_0\mathbf{u}_0 + y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3 + \cdots + y_s\mathbf{u}_s \tag{24}$$

$\mathbf{u}_B = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_s\}$ is called the basis vector, and $\mathbf{y}_s^{\mathbf{T}} = \{y_0, y_1, y_2, y_3, \ldots, y_s\}$ is the vector of unknown coefficients. To calculate the $\mathbf{y}_s$, we define $\mathbf{K}_B = \mathbf{u}_B^{\mathbf{T}}\mathbf{K}\mathbf{u}_B$, $\mathbf{f}_B = \mathbf{u}_B^{\mathbf{T}}\mathbf{f}$, and get the reduced set of $s \times s$ equations $\mathbf{K}_B\mathbf{y} = \mathbf{f}_B$. $\mathbf{y}_s$ is calculated by solving this $s \times s$ equation, then Eq. (24) is called the solution of the Kirsch combined approximation.

Recently, the Padé approximate was used to achieve good accuracy in the static displacement reanalysis. The Padé approximate and the Kirsch combined approximate can give satisfactory solutions for large changes in the structural parameters [12,13].

IV. Epsilon Algorithm for Structural Static Displacement Reanalysis

Now we use the vector epsilon algorithm to develop a new reanalysis method of the modified structure. Assume the solution of Eq. (13) with the following form:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \cdots + \mathbf{u}_s + \cdots \tag{25}$$

where $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_s, \ldots$ can be determined by Eqs. (19–22). We define a vector sequence $\{\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_s, \ldots\}$ where $\mathbf{s}_0 = \mathbf{u}_0$, $\mathbf{s}_1 = \mathbf{u}_0 + \mathbf{u}_1$, $\mathbf{s}_2 = \mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2$, and in general

$$\mathbf{s}_i = \sum_{j=0}^{i}\mathbf{u}_j \qquad i = 0, 1, 2, \ldots, s, \ldots \tag{26}$$

We use the vector epsilon algorithm in the sequence $\{\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_s, \ldots\}$. We have pointed out the epsilon iterative table in Table 1, where the odd row is meaningless, and the even row is the Shanks transformation. The solution is the last even row in the epsilon algorithm shown in Table 1.

$$\mathbf{u} = \boldsymbol{\varepsilon}_{2k}^{(j_0)} \tag{27}$$

where $2k$ is the last even row in Table 1, $j_0 = 0$ or 1.

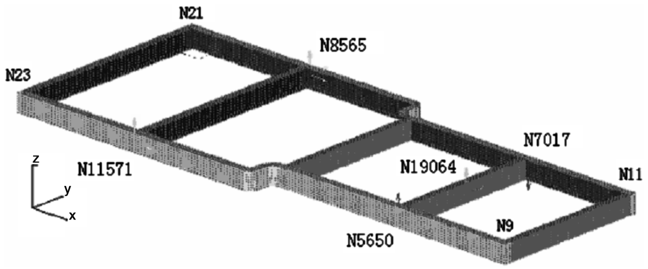

Fig. 1 Chassis structure.

Comparing with the Padé approximate and the Kirsch combined approximate, it can be seen that the epsilon algorithm is used to accelerate the convergence of the perturbation solution other than to use its linear combination or the Padé approximate.

Using the epsilon algorithm, the static displacement reanalysis procedure can be summarized as follows:

1) Calculate the $\mathbf{u}_0$ from Eq. (12).
2) Compute $\mathbf{u}_i$ by Eqs. (20–22).
3) Use Eq. (26) to get $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_s, \ldots$.
4) Form the epsilon-algorithm table by Eqs. (8–11).
5) The solution is evaluated by Eq. (27).

Once the displacements are evaluated, the stresses can be determined explicitly by $\boldsymbol{\sigma} = \mathbf{T}\mathbf{u}$, where $\mathbf{T}$ is the stress transformation matrix.

V. Numerical Example

To illustrate the method, a numerical example of a chassis structure shown in Fig. 1 is given. The finite element model of the given structure consists of 20,953 nodes and 19,880 shell elements. The parameters of the structure are as follows. The Young's modulus is $E = 2.1 \times 10^{11}$ Pa, the mass density is $\rho = 7.8 \times 10^3\ \mathrm{kg/m^3}$, and the thickness of the shell elements is 0.005 m. The chassis is hinged at the nodes 11,771, 8565, and 19,064. Two forces with magnitude $f = 500$ N are applied to node 5650 and node 7017 in the negative Z direction, respectively.

To illustrate results for large changes of parameters, assume that the modifications of parameters are given as follows (t_0 and t_i are thickness):

$$t_0 = 0.005\ \mathrm{m} \qquad t_i = (1 + \beta_i)t_0 \qquad \beta_i = 0.2i \qquad i = 1, 2, 3$$

The results of the z displacements of nodes 9, 11, 21, and 23 obtained by the Kirsch combined approximate, the present epsilon algorithm, and the exact method are listed in Tables 4–6. From the numerical results, it can be seen that both the present epsilon algorithm and the Kirsch combined approximate can give excellent results. For example, in the case of $\beta = 0.6$, $t_3 = 0.008$ m, the error percentage of the epsilon algorithm $E_\varepsilon = |(D_E - D_\varepsilon)/D_E| \times 100$ is 0.045 for $Z11$, and the corresponding value of the Kirsch combined

Table 4 Numerical solutions and comparison of different methods ($\beta_1 = 0.2, t_1 = 0.006$ m)

DOF	Exact, D_E	Kirsch, D_K	Error, E_k, %	Epsilon, D_ε	Error, E_ε, %
$Z9$	3.236E − 02	3.236E − 02	0.000	3.236E − 02	0.000
$Z11$	−3.245E − 02	−3.245E − 02	0.000	−3.245E − 02	0.000
$Z21$	1.172E − 02	1.172E − 02	0.000	1.172E − 02	0.000
$Z23$	−1.151E − 02	−1.152E − 02	0.087	−1.152E − 02	0.087

Table 5 Numerical solutions and comparison of different methods ($\beta_2 = 0.4, t_2 = 0.007$ m)

DOF	Exact, D_E	Kirsch, D_K	Error, E_k, %	Epsilon, D_ε	Error, E_ε, %
$Z9$	2.667E − 02	2.668E − 02	0.037	2.668E − 02	0.037
$Z11$	−2.674E − 02	−2.675E − 02	0.037	−2.675E − 02	0.037
$Z21$	9.771E − 03	9.775E − 03	0.041	9.775E − 03	0.041
$Z23$	−9.607E − 03	−9.611E − 03	0.042	−9.611E − 03	0.042

Table 6 Numerical solutions and comparison of different methods ($\beta_3 = 0.6, t_3 = 0.008$ m)

DOF	Exact, D_E	Kirsch, D_K	Error, E_k, %	Epsilon, D_ε	Error, E_ε, %
Z9	2.206E − 02	2.210E − 02	0.181	2.206E − 02	0.000
Z11	−2.212E − 02	−2.217E − 02	0.226	−2.213E − 02	0.045
Z21	8.174E − 03	8.193E − 03	0.232	8.176E − 03	0.024
Z23	−8.040E − 03	−8.059E − 03	0.236	−8.043E − 03	0.037

Table 7 Computational cost analysis

Kirsch combined approximate		Epsilon algorithm	
Operation	Approximate multiplications	Operation	Approximate multiplications
$\mathbf{K}_B = \mathbf{u}_B^{\mathbf{T}}\mathbf{K}\mathbf{u}_B$	$sn^2 + ns^2$	$\boldsymbol{\varepsilon}_{k+1}^{(j)} = \boldsymbol{\varepsilon}_{k-1}^{(j+1)} + [\boldsymbol{\varepsilon}_k^{(j+1)} - \boldsymbol{\varepsilon}_k^{(j)}]^{-1}$	$2n$ (n additions and n subtractions)
$\mathbf{f}_B = \mathbf{u}_B^{\mathbf{T}}\mathbf{f}$	sn	$\mathbf{u}^{-1} = \frac{\mathbf{u}^*}{(\mathbf{u}^H\mathbf{u})} = \frac{\mathbf{u}^*}{\sum_{i=1}^{d} \lvert u_i \rvert^2}$	$2n$ (n multiplies and n divisions)
Solve $\mathbf{K}_B\mathbf{y} = \mathbf{f}_B$	$\frac{s^3}{3} + s^2$	Epsilon table	$\frac{(s+1)s}{2}$
$\mathbf{u} = \mathbf{u}_B\mathbf{y}_s$	sn	——	——
Total	$sn^2 + (s^2 + 2s)n + \frac{s^3}{3} + s^2$	Total	$s(s+1)n$ multi/divid $s(s+1)n$ add/sub

[a]Note: n is the number of DOF of the structure; s is the number of basis vectors to be used. In this example, $s = 5$. The exact solution is based on the Cholesky decomposition.

Table 8 CPU time (s)

Solution cases	Exact	Kirsch	Epsilon
$\beta_1 = 0.2$	36.10	27.01	11.00
$\beta_2 = 0.4$	35.94	26.89	10.92
$\beta_3 = 0.6$	35.98	26.93	11.06

approximate $E_k = \lvert(D_E - D_K)/D_E\rvert \times 100$ is 0.226; on the other hand, $E_\varepsilon = 0.037$ and $E_k = 0.236$ for $Z23$. The computational cost and CPU time are compared in Tables 7 and 8.

VI. Conclusions

The epsilon algorithm has been developed for static displacement reanalysis. The present method is based on the perturbation solutions and the computational effort is usually much smaller than the effort needed for the full analysis of the modified structure. This method is easiest to implement with a general finite element system and convenient to use in various engineering problems. The numerical example of the chassis structure illustrates that the satisfied approximate results are achieved for large changes in the design. Future study should show that the epsilon algorithm will be very effective in the eigenvalue reanalysis of the modified structures.

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